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NUMERICAL ANALYSIS OF NECKING CONDITIONS IN A THERMOVISCOPLASTIC ROD IN TENSION

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Necking conditions in a thermoviscoplastic rod in tension are studied with allowance for heat transfer within a broad range of strain rates and temperatures. The problem is solved in a long-wave approximation by the methods of linear perturbation analysis. The effect of the amplitude of the perturbations on the stability of plastic deformation is studied by means of nonlinear analysis. The calculations show that: (1) *a neck is formed during tension of a solid rod for the values of the parameters realized in the experiment;* (2) *the critical strain during active tension depends appreciably on the wave number, particularly in the region of small wave numbers;* 3) *the critical strain does not depend significantly on the wave number at low strain rates.*

Introduction. It is known [1, 2] that large plastic strains are associated with the manifestation of strain anomalies: ductility troughs, elongation maxima, and superplasticity. These phenomena are generally seen after the localization of deformation. In the case of rods, strain localization is manifested in the formation of a neck. Two stages are usually recognized in the development of such localization [3]:

(1) the formation of a "running" neck, i.e., the instantaneous initiation and stabilization of local thinnings of the specimen;

(2) upon the subsequent attainment of compressive strain of the order of 0.15–0.25, the appearance of a stable neck that continuously decreases in volume until fracture occurs.

Presnyakov [2] concluded that stable localization begins in connection with the growth of spontaneously forming necks at sufficiently large strains. The effect of the necks on each other makes the process stable.

It is known [1, 2] that the fracture of a rod with the formation of a neck usually begins in the central fibers of the metal, where the contraction from the normal tensile stresses is greatest. However, conflicting results were obtained from later studies of the relationship between strain localization and the accumulation of discontinuities. For example, it was shown in [4] that the critical fragmented structure is formed and nucleated microcracks appear in the specimen only within the necked region and that these events occur long after the neck has been formed. During deformation, the specimen remains continuous for 90–95% of its life. The avalanche emergence of microcracks in the specimen occurs only at the very last moment, and their growth and coalescence results in its almost instantaneous fracture.

On the other hand, it was indicated by Cheremskoi et al. [5] that the multiple formation of nucleated discontinuities generally begins during an early stage of plastic deformation and progresses in proportion to the strain. The intensive formation, growth, and coalescence of discontinuities occurs mainly in thin layers near the surface. The rate of accumulation and concentration of discontinuities in these layers is 1–3 orders greater than in the bulk of the material.

Based on the results reported in [5], Naimark and Ladygin [3] concluded that it is impossible to adequately describe the localization of a plastic flow only in such variables as stress, strain, and strain rate. According to [3], it is also necessary to know the number, distribution, and rate of displacement of the defects in the material. A model of viscous fracture was presented in [6, 7] and accounts simultaneously for

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nucleation and growth of voids. The model makes it possible to calculate the life of a uniformly stressed elementary volume of a solid. However, it does not consider the thermomechanical and geometric properties of the specimen and the test conditions. The contradictions in the experimental data in [4, 5] are too great to presently allow the construction of a rigorous theory of necking in rods in tension.

Previous theoretical studies of the model did not account for the growth of discontinuities during deformation, although satisfactory agreement between experimental and theoretical results was reported in many investigations [see, e.g., [8–11]). Our goal here is to analyze the conditions of the formation of a stable neck in a tensioned rod made of a material with complex rheological properties that is free of discontinuities.

1. Formulation of the Problem. The most general formulation of the problem of necking in a solid rod was given in [12, 13], which examined the plastic deformation of a rod in uniaxial tension. Here, it is assumed that the material of the rod is homogeneous and incompressible and has the density ρ_0. The equation of motion, mass conservation law, compatibility equation, and heat-transfer equation for the given model have the form

$$\rho_0 A_0(x)\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}[\sigma A(t,x)], \quad \frac{\partial \varepsilon}{\partial t} = -\frac{\dot{A}}{A}, \quad \frac{\partial v}{\partial x} = \frac{A(0,x)}{A(t,x)}\frac{\partial \varepsilon}{\partial t}, \quad C\frac{\partial \theta}{\partial t} = k\frac{\partial^2 \theta}{\partial x^2} + \beta\sigma\frac{\partial \varepsilon}{\partial t}, \tag{1.1}$$

where v is the rate of displacement, $A(t,x)$ is the cross-sectional area at the moment of time t; $A_0(x) = A(0,x)$, ε is strain, θ is the temperature, C is the heat capacity; k is the thermal conductivity, and β is the fraction of the plastic work that is converted to heat.

The function $\sigma = F_t^{-1}\psi(\theta,\varepsilon,\dot{\varepsilon})$ models the nonlinear elastoplastic temperature response of the material. Here $F_t^{-1} = (1+2R_c/R)\log(1+R/2R_c)$ is a multiplier accounting for the fact that the stress state in the neck of the rod is triaxial [1], $R(x,t)$ is the local radius of the cross section of the rod, and R_c is the radius of the neck, these two quantities being connected by the relation $R_c^{-1} = (\partial^2 R/\partial x^2)(1+(\partial R/\partial x)^2)^{-3/2}$.

Assuming that the initial cross section of the rod is uniform $A_0(x) = A_0 = \text{const}$, in accordance with the mass conservation law we obtain $A(t,x) = A_0 e^{-\varepsilon}$. Thus, system (1.1), which describes the behavior of the specimen for large plastic strains, has the form

$$\frac{\partial \varepsilon}{\partial t} = e^{-\varepsilon}\frac{\partial v}{\partial x}, \qquad \rho_0\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}[\sigma e^{-\varepsilon}], \qquad C\frac{\partial \theta}{\partial t} = k\frac{\partial^2 \theta}{\partial x^2} + \beta\sigma\frac{\partial \varepsilon}{\partial t}. \tag{1.2}$$

As $\sigma = F_t^{-1}\psi(\theta,\varepsilon,\dot{\varepsilon})$, we examine the relation [12, 13]

$$\sigma = \mu F_t^{-1}\varepsilon^n\dot{\varepsilon}^m\theta^\nu, \tag{1.3}$$

where μ, n, m, and ν are constants.

We adopt the following initial and boundary conditions:

$$\begin{gathered} t=0: \quad \varepsilon = \bar{\varepsilon}_0, \quad \dot{\varepsilon} = \frac{V}{l_0} = \bar{\dot{\varepsilon}}, \quad v = \frac{V}{l_0}x = \bar{\dot{\varepsilon}}x, \quad \theta_0 = \bar{\theta}_0; \\ x=0: \quad v=0, \quad \frac{\partial \theta}{\partial x} = 0; \quad x = l_0: \quad v = V, \quad \frac{\partial \theta}{\partial x} = 0. \end{gathered} \tag{1.4}$$

Here l_0 is the length of the rod and V, $\bar{\theta}_0$, and $\bar{\varepsilon}_0$ are constants.

2. Linear Analysis. We shall examine the homogeneous time-dependent solution $(\varepsilon_0, \sigma_0, v_0, \theta_0, F_{t0})$ of Eqs. (1.2) and (1.3) at the initial moment of time t_0 with the initial and boundary conditions (1.4). We represent the perturbation of this solution in the form

$$\begin{gathered} \varepsilon(x,t) = \varepsilon_0(t) + \delta\varepsilon(x,t) = \varepsilon_0(t) + \delta\varepsilon_0 e^{\eta(t-t_0)}e^{i\xi x}, \\ \sigma(x,t) = \sigma_0(t) + \delta\sigma(x,t) = \sigma_0(t) + \delta\sigma_0 e^{\eta(t-t_0)}e^{i\xi x}, \\ v(x,t) = v_0(t) + \delta v(x,t) = v_0(t) + \delta v_0 e^{\eta(t-t_0)}e^{i\xi x}, \\ \theta(x,t) = \theta_0(t) + \delta\theta(x,t) = \theta_0(t) + \delta\theta_0 e^{\eta(t-t_0)}e^{i\xi x}, \\ F_t(x,t) = F_{t0}(t) + \delta F_t(x,t) = F_{t0}(t) + \delta F_{t0} e^{\eta(t-t_0)}e^{i\xi x}, \end{gathered} \tag{2.1}$$

where the difference $(\delta\varepsilon, \delta\sigma, \delta v, \delta\theta, \delta F_t)$ is assumed to be small compared to $(\varepsilon_0, \sigma_0, v_0, \theta_0, F_{t0})$, $\eta = \delta\dot{\varepsilon}/\delta\varepsilon$, and ξ is the wave number.

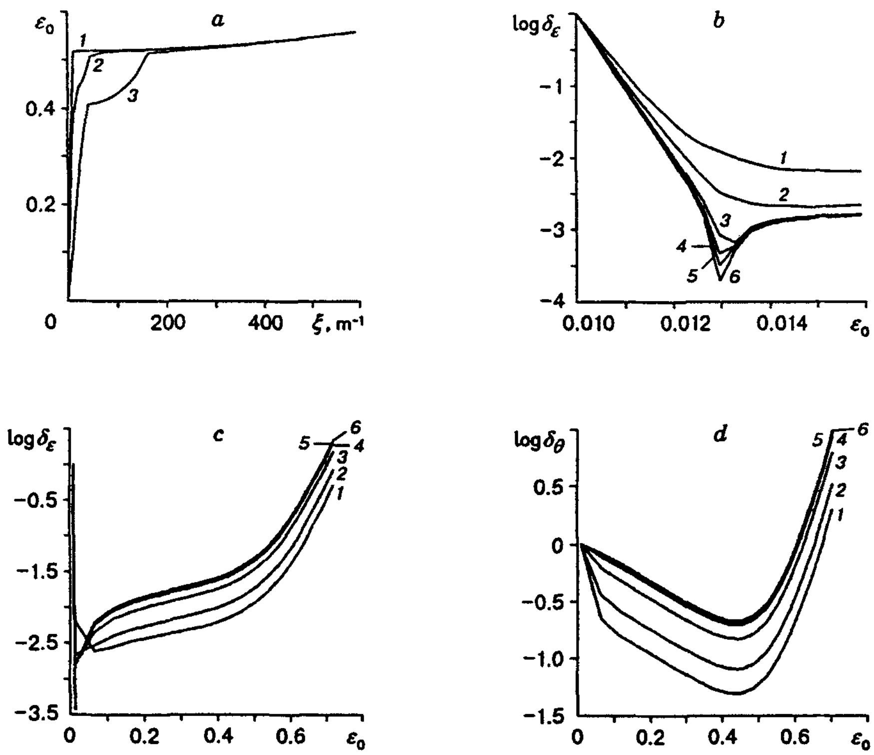

Fig. 1

Substituting (2.1) into (1.2) and considering that $F_{t0} = 1$ and $\delta F_{t0} = -(A_0/2\pi)\xi^2 e^{-2\varepsilon_0}\delta\sigma_0$, we obtain a system of linear equations with the unknown $\delta\varepsilon_0$, $\delta\sigma_0$, δv_0, and $\delta\theta_0$

$$\left(\frac{\partial\psi}{\partial\varepsilon} + \eta\frac{\partial\psi}{\partial\dot{\varepsilon}} + \frac{A_0}{2\pi}\xi^2 e^{-2\varepsilon_0}\sigma_0\right)\delta\varepsilon_0 - \delta\sigma_0 + \frac{\partial\psi}{\partial\theta}\delta\theta_0 = 0,$$

$$-i\xi\sigma_0 e^{-\varepsilon_0}\delta\varepsilon_0 + i\xi e^{-\varepsilon_0}\delta\sigma_0 - \eta\rho_0\delta v_0 = 0, \qquad (\eta + \dot{\varepsilon}_0)\delta\varepsilon_0 - i\xi e^{-\varepsilon_0}\delta v_0 = 0,$$

$$\eta\beta\sigma_0\delta\varepsilon_0 + \beta\dot{\varepsilon}_0\delta\sigma_0 - (C\eta + k\xi^2)\delta\theta_0 = 0.$$

The roots of the characteristic equation of the given system determine the stability of the solution of the problem, which corresponds to uniform tension of the rod (the homogeneous solution).

The characteristic equation for the given system has the form

$$a_0\eta^3 + a_1\eta^2 + a_2\eta + a_3 = 0, \tag{2.2}$$

where

$$a_0 = \rho_0 C; \qquad a_1 = \rho_0 k\xi^2 + C\left(\rho_0\dot{\varepsilon}_0 + \frac{\partial\psi}{\partial\dot{\varepsilon}}\xi^2 e^{-2\varepsilon_0}\right) - \beta\frac{\partial\psi}{\partial\theta}\rho_0\dot{\varepsilon}_0;$$

$$a_2 = k\xi^2\left(\rho_0\dot{\varepsilon}_0 + \frac{\partial\psi}{\partial\dot{\varepsilon}}\xi^2 e^{-2\varepsilon_0}\right) + C\left(\frac{\partial\psi}{\partial\varepsilon} - \sigma_0 + \frac{A_0}{2\pi}\xi^2 e^{-2\varepsilon_0}\sigma_0\right)\xi^2 e^{-2\varepsilon_0} - \beta\frac{\partial\psi}{\partial\theta}\left(\rho_0\dot{\varepsilon}_0^2 - \sigma_0\xi^2 e^{-2\varepsilon_0}\right);$$

$$a_3 = \beta\frac{\partial\psi}{\partial\theta}\dot{\varepsilon}_0\xi^2 e^{-2\varepsilon_0}\sigma_0 + k\left(\frac{\partial\psi}{\partial\varepsilon} - \sigma_0 + \frac{A_0}{2\pi}\xi^2 e^{-2\varepsilon_0}\sigma_0\right)\xi^4 e^{-2\varepsilon_0}.$$

In accordance with the Routh–Hurwitz theory of stability, the solution of the problem will be stable if all of the roots of the characteristic equation of the given linearized system have a negative real part. In accordance with the Leonare–Schipper condition, this will be the case if all the coefficients of polynomial (2.2) are positive and if the condition $a_1a_2 - a_0a_3 > 0$ is satisfied.

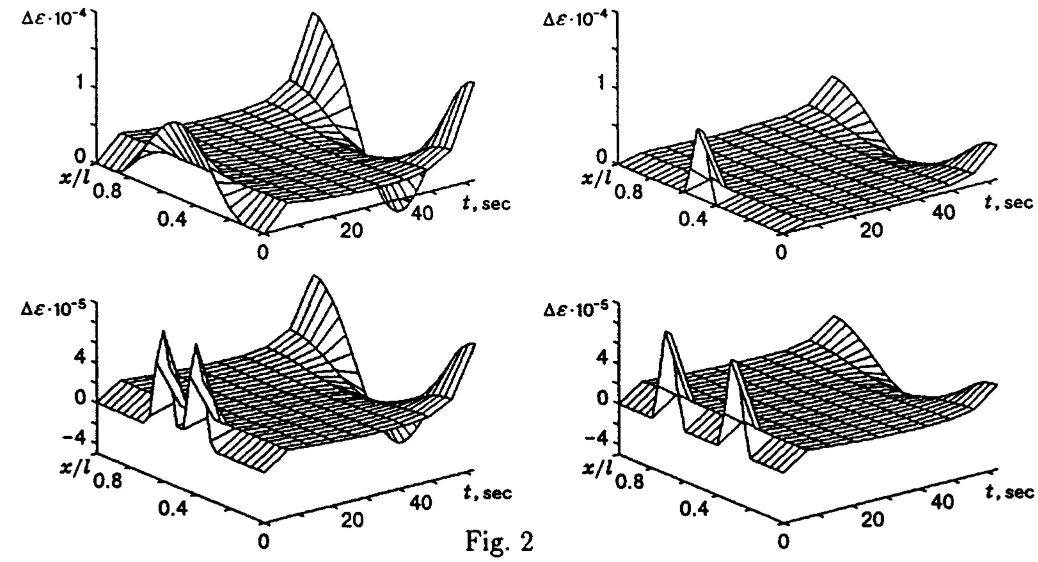

Fig. 2

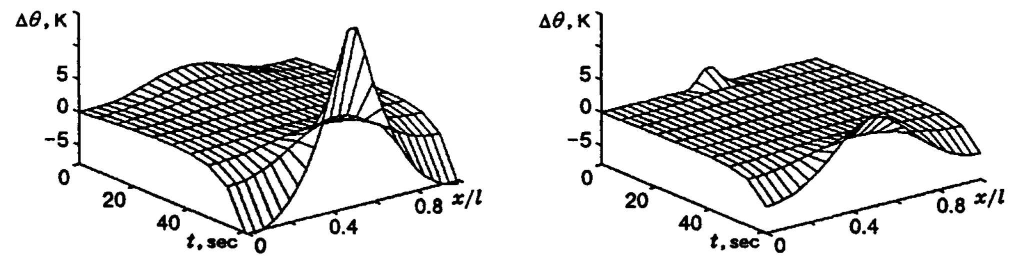

Fig. 3

The following relation was used in calculating $\dot{\varepsilon}_0$ and θ_0:

$$\dot{\varepsilon}_0 = \bar{\dot{\varepsilon}}_0 e^{-\varepsilon_0}, \qquad \theta_0 = \bar{\theta}_0\Big[1 + \frac{(1-\nu)\beta\bar{\dot{\varepsilon}}_0^{\,m}\mu}{C\bar{\theta}_0^{1-\nu}} \int\limits_0^{\varepsilon_0} u^n e^{-mu}\,du\Big]^{1/(1-\nu)}.$$

Figure 1a shows the region of stability of the homogeneous solution of problem (1.2)–(1.4) with $\mu = 2.486 \cdot 10^{10}$, $n = 0.52$, $C = 3.6 \cdot 10^6$ J/(m^3·K), $k = 15$ W/(m·K), $\bar{\theta}_0 = 294$ K, $m = 0.02$, $\nu = -0.5$, $\rho_0 = 7800$ kg/m^3, $A_0 = 4 \cdot 10^{-6}$ m^2 [12] for different deformation rates of the rod $\bar{\dot{\varepsilon}}_0$ (curves 1–3 correspond to the values $\bar{\dot{\varepsilon}}_0 = 1.66 \cdot 10^{-5}$, $1.66 \cdot 10^{-3}$, and $1.66 \cdot 10^{-2}$ sec^{-1}). The solution will be stable against perturbation when ξ and ε are below the respective curves.

3. Nonlinear Analysis. We performed a nonlinear analysis of the stability of the formation of a neck in a tensioned specimen to refine the results of the linear analysis. A strain perturbation of the following form was added to the homogeneous solution of the problem at the moment of time $t = 0$:

$$\varepsilon_p = \varepsilon_0 \delta_0 \sin^2(\xi(x - a)),$$

where δ_0 is the amplitude of the initial perturbation, $\xi = \pi/(b - a)$, and $a < x < b$ (a and b are the coordinates of the left and right boundaries of the perturbed region). The choice of the given type of perturbation is related to the need to compare the results of linear and nonlinear analyses, and it allows us to study the conditions of neck formation in relation to the amplitude of the perturbation δ_0. Figure 1b and c shows the results of calculations of the evolution of the plastic strain (1.1)–(1.3) with perturbed initial conditions for $l_0 = 0.05$ m and $\bar{\dot{\varepsilon}}_0 = 0.01$. The figure shows the dependences of the relative amplitude of the perturbation δ_ε [$\delta_\varepsilon(t) = (\max\limits_x \varepsilon(x,t) - \min\limits_x \varepsilon(x,t))/\delta_0$, where $0 < x < l_0$] on the uniform strain ε_0 at $\delta_0 = 0.01$ for $\xi = 563$, 314, 157, 104.7, 78.5, and 62.8 m^{-1} (curves 1–6). The results for the initial period of deformation corresponding to decay of the perturbation are shown in Fig. 1b, while the results for the period characterized by growth of the perturbation are shown in Fig. 1c. The differences in the behavior of the perturbation are relatively small for $\xi = 62.8$, 78.5, and 104.7 m^{-1}, so that curves 4–6 nearly coincide.

We similarly examined the evolution of a temperature perturbation

$$\theta_p = \theta_0 \delta_0 \sin^2(\xi(x-a)).$$

Figure 1d shows the relative amplitude of the temperature perturbation $\delta_\theta(t) = (\max_x \theta(x,t) - \min_x \theta(x,t))/\delta_0$, where $0 < x < l_0$ for the six values of ξ and δ_0, and $\bar{\varepsilon}_0$ indicated above. Figure 2 shows the strain distribution $\Delta\varepsilon = \varepsilon_p - \varepsilon_0$ over time along the rod for four different modes of initial perturbation ($\bar{\dot{\varepsilon}}_0 = 1.66 \cdot 10^{-2}$ sec^{-1}). Figure 3 shows the temperature distribution $\Delta\theta = \theta_p - \theta_0$ over time for two modes.

4. Analysis of the Results. It can be seen from Fig. 1b and c that the perturbation ε_p initially decays rapidly. The minimum amplitude of the perturbation is lower and is reached more quickly for small ξ than for large ξ. Then δ_ε begins to slowly increase, and rapid growth of the perturbation is seen as the boundary of the stable region is approached (Fig. 1a). Here, the larger the value of ξ, the slower the growth of the perturbation and the later the moment at which the accelerated development of instability is seen. As regards the temperature perturbation θ_p, higher values of ξ correspond to lower values of the minimum perturbations, and their abrupt growth begins later than for lower values of ξ. Calculations performed for different amplitudes of the initial perturbation $10^{-5} < \delta_0 < 10^{-2}$ showed that the results are nearly independent of δ_0. Figures 2 and 3 refine these results, demonstrating that perturbations of any initial shape decay rapidly during the first stage and that $\Delta\varepsilon$ then increases near the ends of the specimen and decreases in its central part. The opposite pattern is seen for the evolution of $\Delta\theta$. We emphasize that Figs. 2 and 3 show results of calculations performed for active tension with a high strain rate $\bar{\dot{\varepsilon}}_0$. It was found that, for small strains $\bar{\dot{\varepsilon}}_0$, the initial perturbations which develop in the stable region completely die out and could not be the reason for the formation of a neck. The critical strain $\bar{\dot{\varepsilon}}_0$ is not significantly dependent on ε_{0c} at sufficiently small ξ (curve 1 in Fig. 1a). Calculations show that a neck is formed in a solid rod subjected to tension and that the conditions of its formation for the chosen material constants of the rod can be realized in an experiment.

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